Lesson 1. Sample Paths

1 Course overview

- In this course, we will learn how to model and analyze systems that evolve dynamically over time and whose behavior is stochastic, or uncertain
- The models we will use to analyze such systems are called **stochastic processes**
- Numerous applications:
  - business and economics – consumer behavior, portfolio management, inventory
  - military – enemy movement, personnel dynamics, maintenance and readiness
  - medicine – disease behavior, policy evaluation
- Techniques used: probability, matrix theory
- This lesson: an example illustrating some basic ideas

2 Sample paths

- A **sample path** is a record of the time-dependent behavior of a system
  - For example, the time that each customer arrives at and departs from the drive-thru at Starbucks
- A sample path can be decomposed into **inputs** and **logic**
  - Inputs = arrival times and service times of customers
  - Logic = how the drive-thru operates: number of windows, first-come-first-served, etc.
- A **simulation** generates new sample paths resulting from changes in the input or logic without building a new system
- **Sample path analysis** uses sample paths generated by a simulation to determine system performance measures

3 Sample path analysis to study submarine behavior

Ballistic missile submarines act as a nuclear deterrence to enemy countries. These submarines move randomly throughout the ocean within a fixed grid assigned by a higher authority. Approximately every 20 minutes, the submarine will turn to change its course in order to clear its *baffles*, the area directly behind the submarine where sonar cannot detect sound.

- Let's consider the movement of a single submarine
- Suppose the submarine operates in an 11×11 grid (see page 5)
- The submarine starts at box $(0, 0)$
- Every 20 minutes, the submarine randomly selects 1 of the 8 cardinal directions (N, NE, E, SE, S, SW, W, NW) and moves one box in that direction
  - For example, after the first 20 minutes, if the submarine selects N, then it will move to box $(0, 1)$

- We want to model the movement of the submarine over time
- Let's model time using indices $n = 0, 1, 2, \ldots$ so that each time index represents

- Let S_n denote the position of the submarine at time n
  - For example, $S_0 =$

- Using an 8-sided die, generate a sample path representing the movement of the submarine for the first 100 minutes, or 5 time steps (see page 5)
  - Repeat this 5 times and record your submarine's 5 final locations
- Did you generate the same sample path multiple times?

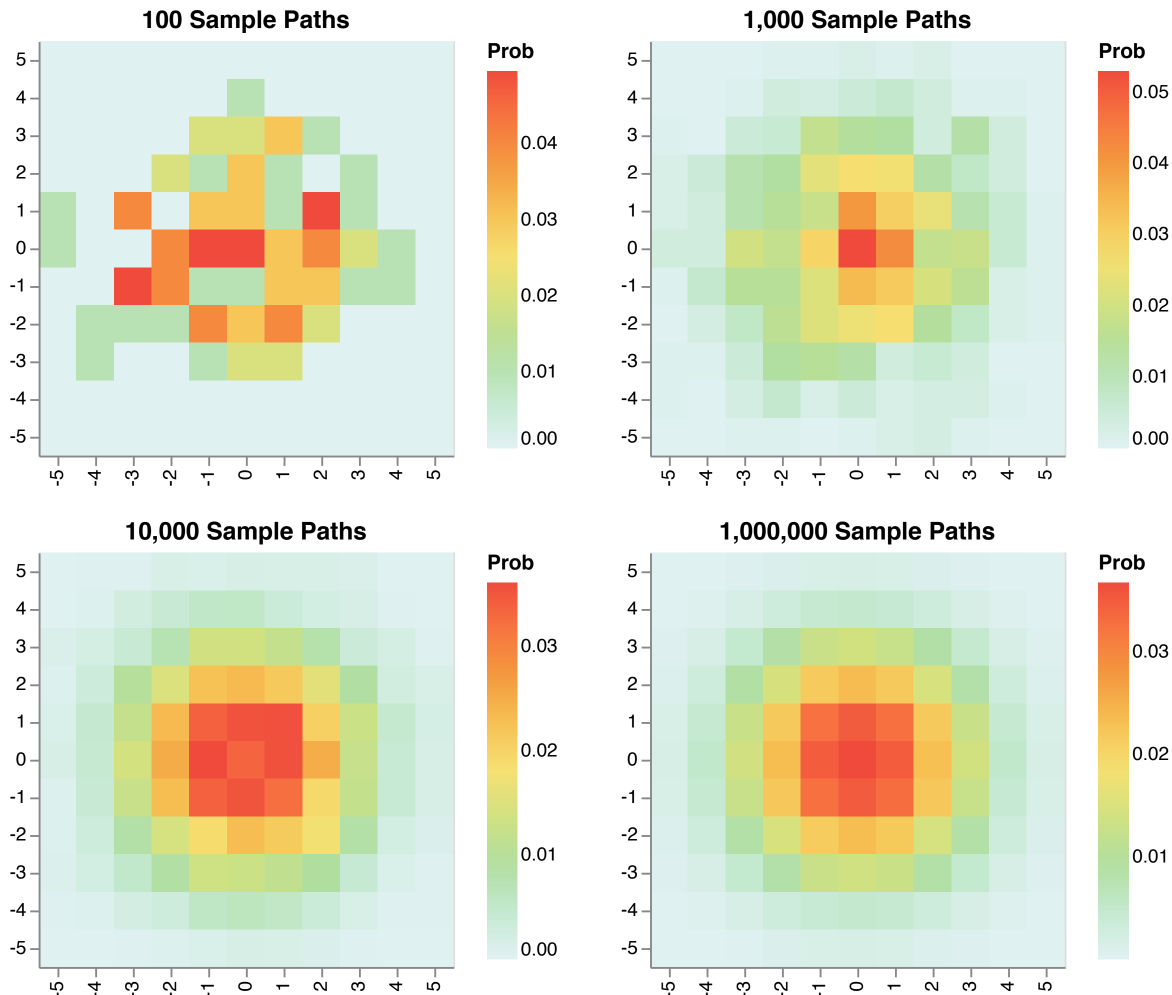
- Assume that you have a fair die. What is the probability that the submarine moves N on the first roll?

- Based on the sample paths you generated, what is the probability that the submarine moves N on the first roll? (In other words, what percentage of time did the submarine move N on the first roll in your simulations?)

- How many possible sample paths can be generated?

- In general, what kind of information can you learn from generating many sample paths?

- Suppose we ran this same experiment, but with many more repetitions – for example, generating
    - 100 sample paths
    - 1,000 sample paths
    - 10,000 sample paths
    - 1,000,000 sample paths
- We would get something that looks like this:



- What do you observe in the probability distributions above?

4 What's next?

- In this lesson's example...
  - We used simulation to generate sample paths
  - We analyzed the sample paths to determine an **empirical probability distribution** for the state of the system – i.e., the location of the submarine after 100 minutes
    - An empirical probability distribution is based on the frequency of observed data
  - Simulation modeling and analysis is the focus of SA421
- If we make certain reasonable assumptions....
  - We can mathematically analyze sample paths without actually generating them to determine a **theoretical probability distribution** for the state of the system
    - A theoretical probability distribution is based on logic or mathematical formulas
  - This kind of modeling and analysis is the focus of this course
- Some examples of models we will study:
  - Markov chains
  - Poisson processes
  - Markov processes and queueing systems
- Such models are used in real-world applications
  - For example, the Undersea Warfighting Development Center in Groton, CT uses Markov chains to model the movement of submarines

Roll the 8-sided die to generate 5 sample paths representing the submarine's movements for the first 100 minutes. Use the following key to determine which cardinal direction the submarine should move.

1	2	3	4	5	6	7	8
N	NE	E	SE	S	SW	W	NW

(−5, 5)	(−4, 5)	(−3, 5)	(−2, 5)	(−1, 5)	(0, 5)	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)
(−5, 4)	(−4, 4)	(−3, 4)	(−2, 4)	(−1, 4)	(0, 4)	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)
(−5, 3)	(−4, 3)	(−3, 3)	(−2, 3)	(−1, 3)	(0, 3)	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)
(−5, 2)	(−4, 2)	(−3, 2)	(−2, 2)	(−1, 2)	(0, 2)	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)
(−5, 1)	(−4, 1)	(−3, 1)	(−2, 1)	(−1, 1)	(0, 1)	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)
(−5, 0)	(−4, 0)	(−3, 0)	(−2, 0)	(−1, 0)	(0, 0)	(1, 0)	(2, 0)	(3, 0)	(4, 0)	(5, 0)
(−5, −1)	(−4, −1)	(−3, −1)	(−2, −1)	(−1, −1)	(0, −1)	(1, −1)	(2, −1)	(3, −1)	(4, −1)	(5, −1)
(−5, −2)	(−4, −2)	(−3, −2)	(−2, −2)	(−1, −2)	(0, −2)	(1, −2)	(2, −2)	(3, −2)	(4, −2)	(5, −2)
(−5, −3)	(−4, −3)	(−3, −3)	(−2, −3)	(−1, −3)	(0, −3)	(1, −3)	(2, −3)	(3, −3)	(4, −3)	(5, −3)
(−5, −4)	(−4, −4)	(−3, −4)	(−2, −4)	(−1, −4)	(0, −4)	(1, −4)	(2, −4)	(3, −4)	(4, −4)	(5, −4)
(−5, −5)	(−4, −5)	(−3, −5)	(−2, −5)	(−1, −5)	(0, −5)	(1, −5)	(2, −5)	(3, −5)	(4, −5)	(5, −5)